

THE COMULTIPLICATION OF MODIFIED QUANTUM AFFINE \mathfrak{sl}_n

QIANG FU

ABSTRACT. Let $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ be the modified quantum affine \mathfrak{sl}_n and let $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ be the positive part of quantum affine \mathfrak{sl}_N . Let $\dot{\mathbf{B}}(n)$ be the canonical basis of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ and let $\mathbf{B}(N)^{\text{ap}}$ be the canonical basis of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$. It is proved in [4] that each structure constant for the multiplication with respect to $\dot{\mathbf{B}}(n)$ coincide with a certain structure constant for the multiplication with respect to $\mathbf{B}(N)^{\text{ap}}$ for $n < N$. In this paper we use the theory of affine quantum Schur algebras to prove that the structure constants for the comultiplication with respect to $\dot{\mathbf{B}}(n)$ are determined by the structure constants for the comultiplication with respect to $\mathbf{B}(N)^{\text{ap}}$ for $n < N$. In particular, the positivity property for the comultiplication of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ follows from the positivity property for the comultiplication of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$.

1. INTRODUCTION

In [7], Grojnowski gave a geometric construction of the comultiplication Δ for quantum group of type A . As a consequence, he obtain that it has positive structure constants with respect to the canonical basis for quantum group of type A (see also [8]). The geometric description of Δ was generalized to the affine case by Lusztig [14].

Let $\mathcal{S}_\Delta(n, r)$ be the affine quantum Schur algebra over $\mathbb{Q}(v)$ (see [5, 6, 13]). Let $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ be the quantum affine \mathfrak{sl}_n . The algebra $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ and $\mathcal{S}_\Delta(n, r)$ are related by an algebra homomorphism $\zeta_r : \mathbf{U}(\widehat{\mathfrak{sl}}_n) \rightarrow \mathcal{S}_\Delta(n, r)$ (see Ginzburg–Vasserot [5] and Lusztig [13]). The map ζ_r can be extended to a surjective algebra homomorphism from $\mathfrak{D}_\Delta(n)$ to $\mathcal{S}_\Delta(n, r)$, where $\mathfrak{D}_\Delta(n)$ is the double Ringel–Hall algebra of affine type A (see [1, 3.8.1]).

It is well known that the positive part $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ has a canonical basis $\mathbf{B}(n)^{\text{ap}}$ with remarkable properties (see Kashiwara [9], Lusztig [10]). Let $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ be the modified form of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$. The algebra $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ is an associative algebra without unity and the category of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ -modules of type 1 is equivalent to the category of unital $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ -modules. The algebra $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ has a canonical basis $\dot{\mathbf{B}}(n)$ constructed by Lusztig [12]. Let $\mathbf{B}(n, r)$ be the canonical basis of $\mathcal{S}_\Delta(n, r)$ (see Lusztig [13]). The compatibility of $\dot{\mathbf{B}}(n)$ and $\mathbf{B}(n, r)$ was proved by Schiffmann–Vasserot [17].

In [4], some good relations among the structure constants for the multiplication with respect to the canonical bases of the three algebras $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$, $\mathcal{S}_\Delta(n, r)$ and $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$ were established.

[†] Supported by the National Natural Science Foundation of China.

In this paper we prove that there are similar relations among the structure constants for the comultiplication with respect to the canonical bases of these algebras. More precisely, we prove in 3.8 that the structure constants for the comultiplication with respect to $\mathbf{B}(n, r)$ are determined by that with respect to $\mathbf{B}(N)^{\text{ap}}$ for $n < N$. Using 3.8, we prove in 4.1 that the structure constants for the comultiplication with respect to $\dot{\mathbf{B}}(n)$ are determined by that with respect to $\mathbf{B}(N)^{\text{ap}}$ for $n < N$. In particular the positivity property for the comultiplication with respect to $\dot{\mathbf{B}}(n)$ follows from the positivity property for the comultiplication with respect to $\mathbf{B}(N)^{\text{ap}}$.

Notation: For a positive integer n , let $\Theta_\Delta(n)$ be the set of all matrices $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ with $a_{i,j} \in \mathbb{N}$ (resp. $a_{i,j} \in \mathbb{Z}$, $a_{i,j} \geq 0$ for all $i \neq j$) such that

- (a) $a_{i,j} = a_{i+n,j+n}$ for $i, j \in \mathbb{Z}$;
- (b) for every $i \in \mathbb{Z}$, both sets $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$ and $\{j \in \mathbb{Z} \mid a_{j,i} \neq 0\}$ are finite.

Let $\Theta_\Delta^+(n) = \{A \in \Theta_\Delta(n) \mid a_{i,j} = 0 \text{ for } i \geq j\}$. For $r \geq 0$, let $\Theta_\Delta(n, r) = \{A \in \Theta_\Delta(n) \mid \sigma(A) = r\}$, where $\sigma(A) = \sum_{1 \leq i \leq n, j \in \mathbb{Z}} a_{i,j}$.

Let $\mathbb{Z}_\Delta^n = \{(\lambda_i)_{i \in \mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}\}$ and $\mathbb{N}_\Delta^n = \{(\lambda_i)_{i \in \mathbb{Z}} \in \mathbb{Z}_\Delta^n \mid \lambda_i \geq 0 \text{ for } i \in \mathbb{Z}\}$. \mathbb{Z}_Δ^n has a natural structure of abelian group. For $r \geq 0$ let $\Lambda_\Delta(n, r) = \{\lambda \in \mathbb{N}_\Delta^n \mid \sigma(\lambda) = r\}$, where $\sigma(\lambda) = \sum_{1 \leq i \leq n} \lambda_i$.

Let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$, where v is an indeterminate. For $c \in \mathbb{Z}$ and $a \in \mathbb{N}$ let $\begin{bmatrix} c \\ a \end{bmatrix} = \prod_{s=1}^a \frac{v^{c-s+1} - v^{-c+s-1}}{v^s - v^{-s}}$.

2. PRELIMINARIES

Let $I = \mathbb{Z}/n\mathbb{Z} = \{1, 2, \dots, n\}$ and let $(c_{i,j})_{i,j \in I}$ be the Cartan matrix of affine type A . Let $\mathfrak{D}_\Delta(n)$ be the double Ringel–Hall algebra of affine type A . The algebra $\mathfrak{D}_\Delta(n)$ is isomorphic to the quantum loop algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ (see [1, 2.5.3]). By [1, 2.3.1 and 2.3.5] we have the following result.

Proposition 2.1. *The algebra $\mathfrak{D}_\Delta(n)$ is the $\mathbb{Q}(v)$ -algebra generated by $E_i, F_i, K_i, K_i^{-1}, \mathbf{z}_s^+, \mathbf{z}_s^-$, for $i \in I, s \in \mathbb{Z}^+$, and relations:*

- (QGL1) $K_i K_j = K_j K_i, K_i K_i^{-1} = 1$;
- (QGL2) $K_i E_j = v^{\delta_{i,j} - \delta_{i,j+1}} E_j K_i, K_i F_j = v^{-\delta_{i,j} + \delta_{i,j+1}} F_j K_i$;
- (QGL3) $E_i F_j - F_j E_i = \delta_{i,j} \frac{\tilde{K}_i - \tilde{K}_i^{-1}}{v - v^{-1}}$, where $\tilde{K}_i = K_i K_{i+1}^{-1}$ (and $\tilde{K}_n = K_n K_1^{-1}$);
- (QGL4) $\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix} E_i^a E_j E_i^b = 0$ for $i \neq j$;
- (QGL5) $\sum_{a+b=1-c_{i,j}} (-1)^a \begin{bmatrix} 1 - c_{i,j} \\ a \end{bmatrix} F_i^a F_j F_i^b = 0$ for $i \neq j$;
- (QGL6) \mathbf{z}_s^+ and \mathbf{z}_s^- are central elements in $\mathfrak{D}_\Delta(n)$.

where $i, j \in I$ and $s, t \in \mathbb{Z}^+$. It is a Hopf algebra with comultiplication Δ defined by

$$\begin{aligned}\Delta(E_i) &= E_i \otimes \tilde{K}_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + \tilde{K}_i^{-1} \otimes F_i, \\ \Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(z_s^{\pm}) = z_s^{\pm} \otimes 1 + 1 \otimes z_s^{\pm};\end{aligned}$$

where $i \in I$ and $s \in \mathbb{Z}^+$.

The extended affine Hecke algebra $\mathcal{H}_\Delta(r)$ is defined to be the $\mathbb{Q}(v)$ -algebra generated by T_i , $X_j^{\pm 1}$ ($1 \leq i \leq r-1$, $1 \leq j \leq r$), and relations

$$\begin{aligned}(T_i + 1)(T_i - v^2) &= 0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad (|i - j| > 1), \\ X_i X_i^{-1} &= 1 = X_i^{-1} X_i, \quad X_i X_j = X_j X_i, \quad T_i X_i T_i = v^2 X_{i+1}, \quad X_j T_i = T_i X_j \quad (j \neq i, i+1).\end{aligned}$$

Let Ω be a vector space over $\mathbb{Q}(v)$ with basis $\{\omega_i \mid i \in \mathbb{Z}\}$. Let $I(n, r) = \{(i_1, \dots, i_r) \in \mathbb{Z}^r \mid 1 \leq i_k \leq n, \forall k\}$. For $\mathbf{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$, write $\omega_{\mathbf{i}} = \omega_{i_1} \otimes \omega_{i_2} \otimes \dots \otimes \omega_{i_r} \in \Omega^{\otimes r}$. The tensor space $\Omega^{\otimes r}$ admits a right $\mathcal{H}_\Delta(r)$ -module structure defined by

$$\begin{cases} \omega_{\mathbf{i}} \cdot X_t^{-1} = \omega_{(i_1, \dots, i_{t-1}, i_t+n, i_{t+1}, \dots, i_r)}, & \text{for all } \mathbf{i} \in \mathbb{Z}^r; \\ \omega_{\mathbf{i}} \cdot T_k = \begin{cases} v^2 \omega_{\mathbf{i}}, & \text{if } i_k = i_{k+1}; \\ v \omega_{(i_1, \dots, i_{k+1}, i_k, \dots, i_r)}, & \text{if } i_k < i_{k+1}; \\ v \omega_{(i_1, \dots, i_{k+1}, i_k, \dots, i_r)} + (v^2 - 1) \omega_{\mathbf{i}}, & \text{if } i_{k+1} < i_k, \end{cases} & \text{for all } \mathbf{i} \in I(n, r), \end{cases}$$

where $1 \leq k \leq r-1$ and $1 \leq t \leq r$ (cf. [18]). The algebra

$$\mathcal{S}_\Delta(n, r) := \text{End}_{\mathcal{H}_\Delta(r)}(\Omega^{\otimes r})$$

is called an affine quantum Schur algebra.

For $\lambda \in \Lambda_\Delta(n, r)$, let $\mathfrak{S}_\lambda := \mathfrak{S}_{(\lambda_1, \dots, \lambda_n)}$ be the corresponding standard Young subgroup of the symmetric group \mathfrak{S}_r . For $\lambda \in \Lambda_\Delta(n, r)$ let $x_\lambda = v^{\ell(w_{0,\lambda})} \omega_{\mathbf{i}_\lambda}$, where $w_{0,\lambda}$ is the longest element in \mathfrak{S}_λ and

$$\mathbf{i}_\lambda = (\underbrace{1, \dots, 1}_{\lambda_1}, \underbrace{2, \dots, 2}_{\lambda_2}, \dots, \underbrace{n, \dots, n}_{\lambda_n}) \in I(n, r).$$

Then we have $\Omega^{\otimes r} = \bigoplus_{\lambda \in \Lambda_\Delta(n, r)} x_\lambda \mathcal{H}_\Delta(r)$.

The vector space Ω is a natural $\mathfrak{D}_\Delta(n)$ -module with the action $E_i \cdot \omega_s = \delta_{i+1, \bar{s}} \omega_{s-1}$, $F_i \cdot \omega_s = \delta_{i, \bar{s}} \omega_{s+1}$, $K_i^{\pm 1} \cdot \omega_s = v^{\pm \delta_{i, \bar{s}}} \omega_s$, $z_t^+ \cdot \omega_s = \omega_{s+tn}$, and $z_t^- \cdot \omega_s = \omega_{s-tn}$, where \bar{s} is the integer s modulo n . The Hopf algebra structure of $\mathfrak{D}_\Delta(n)$ induces a $\mathfrak{D}_\Delta(n)$ -module $\Omega^{\otimes r}$. We denote by $\zeta_r : \mathfrak{D}_\Delta(n) \rightarrow \text{End}(\Omega^{\otimes r})$ the corresponding representation. By [1, 3.8.1] we have $\zeta_r(\mathfrak{D}_\Delta(n)) = \mathcal{S}_\Delta(n, r)$.

For $r', r'' \in \mathbb{N}$, there is a natural injective algebra homomorphism

$$\varphi_{r', r''} : \text{End}(\Omega^{\otimes r'}) \otimes \text{End}(\Omega^{\otimes r''}) \rightarrow \text{End}(\Omega^{\otimes r' + r''})$$

such that $\varphi_{r',r''}(f \otimes g)(w_1 \otimes w_2) = f(w_1) \otimes g(w_2)$ for $f \in \text{End}(\Omega^{\otimes r'})$, $g \in \text{End}(\Omega^{\otimes r''})$, $w_1 \in \Omega^{\otimes r'}$, $w_2 \in \Omega^{\otimes r''}$. By restricting the map $\varphi_{r',r''}$ to $\mathcal{S}_\Delta(n, r') \otimes \mathcal{S}_\Delta(n, r'')$, we obtain an algebra isomorphism

$$\varphi_{r',r''} : \mathcal{S}_\Delta(n, r') \otimes \mathcal{S}_\Delta(n, r'') \rightarrow \varphi_{r',r''}(\mathcal{S}_\Delta(n, r') \otimes \mathcal{S}_\Delta(n, r'')).$$

It is clear that we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{D}_\Delta(n) & \xrightarrow{\Delta} & \mathfrak{D}_\Delta(n) \otimes \mathfrak{D}_\Delta(n) \\ \zeta_{r'+r''} \downarrow & & \downarrow \zeta_{r'} \otimes \zeta_{r''} \\ \varphi_{r',r''}(\mathcal{S}_\Delta(n, r') \otimes \mathcal{S}_\Delta(n, r'')) & \xleftarrow{\varphi_{r',r''}} & \mathcal{S}_\Delta(n, r') \otimes \mathcal{S}_\Delta(n, r''). \end{array}$$

So $\mathcal{S}_\Delta(n, r' + r'') = \zeta_{r'+r''}(\mathfrak{D}_\Delta(n)) \subseteq \varphi_{r',r''}(\mathcal{S}_\Delta(n, r') \otimes \mathcal{S}_\Delta(n, r''))$. By restricting $\varphi_{r'+r''}^{-1}$ to $\mathcal{S}_\Delta(n, r' + r'')$, we obtain an algebra homomorphism

$$\Delta_{r',r''} := \varphi_{r'+r''}^{-1} : \mathcal{S}_\Delta(n, r' + r'') \rightarrow \mathcal{S}_\Delta(n, r') \otimes \mathcal{S}_\Delta(n, r'').$$

3. THE CONNECTION BETWEEN $\mathbf{B}(n, r)$ AND $\mathbf{B}(N)^{\text{ap}}$

Let $\mathfrak{D}_\Delta^+(n)$ be the $\mathbb{Q}(v)$ -subalgebra of $\mathfrak{D}_\Delta(n)$ generated by E_i and \mathbf{z}_s^+ for $i \in I$ and $s \in \mathbb{Z}^+$. Let $\mathbf{B}(n) := \{\theta_A^+ \mid A \in \Theta_\Delta^+(n)\}$ be the canonical basis of $\mathfrak{D}_\Delta^+(n)$ (see [18]). For $A = (a_{i,j}) \in \Theta_\Delta^+(n)$ let $\mathbf{d}(A) = (\sum_{s \leq i, t \geq i+1} a_{s,t})_{i \in \mathbb{Z}} \in \mathbb{N}_\Delta^n$. For $\mathbf{j} \in \mathbb{Z}_\Delta^n$ let $\tilde{K}^{\mathbf{j}} = \prod_{1 \leq i \leq n} (\tilde{K}_i)^{j_i}$. For $A, B \in \Theta_\Delta^+(n)$ we write

$$(3.1) \quad \Delta(\theta_A^+) = \sum_{B, C \in \Theta_\Delta^+(n)} \mathbf{f}_{A,B,C} \theta_B^+ \otimes \theta_C^+ \tilde{K}^{\mathbf{d}(B)},$$

where $\mathbf{f}_{A,B,C} \in \mathbb{Z}$. Note that if $\mathbf{f}_{A,B,C} \neq 0$ then $\mathbf{d}(A) = \mathbf{d}(B) + \mathbf{d}(C)$.

A matrix $A = (a_{i,j}) \in \Theta_\Delta(n)$ is said to be aperiodic if for every integer $l \neq 0$ there exists $1 \leq i \leq n$ such that $a_{i,i+l} = 0$. Let $\Theta_\Delta(n)^{\text{ap}}$ be the set of all aperiodic matrices in $\Theta_\Delta(n)$. Let $\Theta_\Delta^+(n)^{\text{ap}} = \Theta_\Delta^+(n) \cap \Theta_\Delta(n)^{\text{ap}}$.

Let $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ be the $\mathbb{Q}(v)$ -subalgebra of $\mathfrak{D}_\Delta(n)$ generated by the elements E_i , F_i and $\tilde{K}_i^{\pm 1}$ for $i \in I$. Let $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$ be the $\mathbb{Q}(v)$ -subalgebra of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ generated by the elements E_i for $i \in I$. Then $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$ is isomorphic to the composition algebra of the cyclic quiver $\Delta(n)$ (see Ringel [16]). Let

$$\mathbf{B}(n)^{\text{ap}} := \{\theta_A^+ \mid A \in \Theta_\Delta^+(n)^{\text{ap}}\}.$$

Then by [11], we know that the set $\mathbf{B}(n)^{\text{ap}}$ forms a $\mathbb{Q}(v)$ -basis for $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$ and is called the canonical basis of $\mathbf{U}(\widehat{\mathfrak{sl}}_n)^+$. The following result are due to Lusztig (see [10] and [12, 14.4.13]).

Theorem 3.1. *For $A, B, C \in \Theta_\Delta^+(n)^{\text{ap}}$ we have $\mathbf{f}_{A,B,C} \in \mathbb{N}[v, v^{-1}]$.*

Let $\mathfrak{S}_{\Delta, r}$ be the group consisting of all permutations $w : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $w(i+r) = w(i) + r$ for $i \in \mathbb{Z}$. For $\lambda, \mu \in \Lambda_{\Delta}(n, r)$, let $\mathcal{D}_{\lambda}^{\Delta} = \{d \mid d \in \mathfrak{S}_{\Delta, r}, \ell(wd) = \ell(w) + \ell(d) \text{ for } w \in \mathfrak{S}_{\lambda}\}$ and $\mathcal{D}_{\lambda, \mu}^{\Delta} = \mathcal{D}_{\lambda}^{\Delta} \cap \mathcal{D}_{\mu}^{\Delta^{-1}}$. For $\lambda \in \Lambda_{\Delta}(n, r)$, $1 \leq i \leq n$ and $k \in \mathbb{Z}$ let $R_{i+kn}^{\lambda} = \{\lambda_{k, i-1} + 1, \lambda_{k, i-1} + 2, \dots, \lambda_{k, i-1} + \lambda_i = \lambda_{k, i}\}$, where $\lambda_{k, i-1} = kr + \sum_{1 \leq t \leq i-1} \lambda_t$. By [18, 7.4] (see also [2, 9.2]), there is a bijective map

$$(3.2) \quad j_{\Delta} : \{(\lambda, d, \mu) \mid d \in \mathcal{D}_{\lambda, \mu}^{\Delta}, \lambda, \mu \in \Lambda_{\Delta}(n, r)\} \longrightarrow \Theta_{\Delta}(n, r)$$

sending (λ, d, μ) to the matrix $A = (|R_k^{\lambda} \cap dR_l^{\mu}|)_{k, l \in \mathbb{Z}}$.

The algebra $\mathcal{S}_{\Delta}(n, r)$ has a normalized $\mathbb{Q}(v)$ -basis $\{[A] \mid A \in \Theta_{\Delta}(n, r)\}$ (cf. [13, 1.9]). Let

$$\mathbf{B}(n, r) := \{\theta_{A, r} \mid A \in \Theta_{\Delta}(n, r)\}$$

be the canonical basis of $\mathcal{S}_{\Delta}(n, r)$ defined by Lusztig [13]. For $\lambda, \mu \in \Lambda_{\Delta}(n, r)$ and $d \in \mathcal{D}_{\lambda, \mu}^{\Delta}$ let $\theta_{\lambda, \mu}^d = \theta_{A, r}$, where $A = j_{\Delta}(\lambda, d, \mu)$. Let ρ be the permutation of \mathbb{Z} sending j to $j+1$ for all $j \in \mathbb{Z}$. Then for $\lambda \in \Lambda_{\Delta}(n, r)$ and $m \in \mathbb{Z}$ we have $\theta_{\lambda, \lambda}^{\rho^{mr}} = [j_{\Delta}(\lambda, \rho^{mr}, \lambda)]$.

Assume $r = r' + r''$ with $r', r'' \in \mathbb{N}$. For $A, B \in \Theta_{\Delta}(n, r)$ we write

$$(3.3) \quad \Delta_{r', r''}(\theta_{A, r}) = \sum_{\substack{B \in \Theta_{\Delta}(n, r') \\ C \in \Theta_{\Delta}(n, r'')}} \mathfrak{g}_{A, B, C}^{r', r''} \theta_{B, r'} \otimes \theta_{C, r''}$$

where $\mathfrak{g}_{A, B, C}^{r', r''} \in \mathbb{Q}(v)$.

Let $T_{\rho} = X_1^{-1} \tilde{T}_1^{-1} \cdots \tilde{T}_{r-1}^{-1} \in \mathcal{H}_{\Delta}(r)$, where $\tilde{T}_i = v^{-1} T_i$. We are now ready to compute $\Delta_{r', r''}(\theta_{\lambda, \lambda}^{\rho^{mr}})$. We need the following lemma.

Lemma 3.2. *For $1 \leq k \leq r$, we have $(\tilde{T}_{k-1} \cdots \tilde{T}_2 \tilde{T}_1)^k = X_1 X_2 \cdots X_k$. In particular, we have $T_{\rho}^r = X_1^{-1} X_2^{-1} \cdots X_r^{-1}$.*

Proof. We apply induction on k . The case $k = 1, 2$ is trivial. We now assume $k > 2$. For $1 \leq s \leq k-1$ let $\mathbf{x}_{s, k} = (\tilde{T}_{k-1} \cdots \tilde{T}_1 X_1)^{k-s} (\tilde{T}_{k-s} \cdots \tilde{T}_{k-2} \tilde{T}_{k-1}) (\tilde{T}_{k-2} \cdots \tilde{T}_1 X_1)^s$. For $1 \leq s \leq k-2$ we have $\mathbf{x}_{s, k} = (\tilde{T}_{k-1} \cdots \tilde{T}_1 X_1)^{k-s-1} \mathbf{y}_{s, k} (\tilde{T}_{k-s-2} \cdots \tilde{T}_1 X_1) (\tilde{T}_{k-2} \cdots \tilde{T}_1 X_1)^s$ where $\mathbf{y}_{s, k} = \tilde{T}_{k-1} \cdots \tilde{T}_{k-s} \tilde{T}_{k-s-1} \tilde{T}_{k-s} \cdots \tilde{T}_{k-1}$. Since $\mathbf{y}_{s, k} = \tilde{T}_{k-s-1} \mathbf{y}_{s-1, k} \tilde{T}_{k-s-1}$ we have $\mathbf{y}_{s, k} = \tilde{T}_{k-s-1} \cdots \tilde{T}_{k-2} \tilde{T}_{k-1} \tilde{T}_{k-2} \cdots \tilde{T}_{k-s-1}$. It follows that

$$\mathbf{x}_{s, k} = (\tilde{T}_{k-1} \cdots \tilde{T}_1 X_1)^{k-s-1} (\tilde{T}_{k-s-1} \cdots \tilde{T}_{k-2} \tilde{T}_{k-1}) (\tilde{T}_{k-2} \cdots \tilde{T}_1 X_1)^{s+1} = \mathbf{x}_{s+1, k}$$

for $1 \leq s \leq k-2$. Consequently, by induction we have $(\tilde{T}_{k-1} \cdots \tilde{T}_2 \tilde{T}_1)^k = \mathbf{x}_{1, k} = \mathbf{x}_{k-1, k} = (\tilde{T}_{k-1} \cdots \tilde{T}_1 X_1) (\tilde{T}_1 \tilde{T}_2 \cdots \tilde{T}_{k-1}) (\tilde{T}_{k-2} \cdots \tilde{T}_2 \tilde{T}_1)^{k-1} = X_k \cdots X_2 X_1$. \square

Assume $r = r' + r''$ with $r', r'' \in \mathbb{N}$. There is an injective algebra homomorphism

$$\kappa_{r', r''} : \mathcal{H}_{\Delta}(r') \otimes \mathcal{H}_{\Delta}(r'') \rightarrow \mathcal{H}_{\Delta}(r)$$

such that $\kappa_{r',r''}(T_i \otimes 1) = T_i$ ($1 \leq i \leq r' - 1$), $\kappa_{r',r''}(X_j \otimes 1) = X_j$ ($1 \leq j \leq r'$), $\kappa_{r',r''}(1 \otimes T_i) = T_{r'+i}$ ($1 \leq i \leq r'' - 1$), $\kappa_{r',r''}(1 \otimes X_j) = X_{r'+j}$ ($1 \leq j \leq r''$). We will indentify $\mathcal{H}_\Delta(r') \otimes \mathcal{H}_\Delta(r'')$ as a subalgebra of $\mathcal{H}_\Delta(r)$ via $\kappa_{r',r''}$.

Lemma 3.3. *Assume $r = r' + r''$ with $r', r'' \in \mathbb{N}$. For $\lambda \in \Lambda_\Delta(n, r)$ and $m \in \mathbb{Z}$ we have*

$$\Delta_{r',r''}(\theta_{\lambda,\lambda}^{\rho^{mr}}) = \sum_{\substack{\alpha \in \Lambda_\Delta(n,r'), \beta \in \Lambda_\Delta(n,r'') \\ \lambda = \alpha + \beta}} \theta_{\alpha,\alpha}^{\rho^{mr'}} \otimes \theta_{\beta,\beta}^{\rho^{mr''}}.$$

Proof. Note that $\theta_{\lambda,\lambda}^{\rho^{mr}}(x_\mu h) = \delta_{\lambda,\mu} x_\mu h T_{\rho^{mr}}$ for $\mu \in \Lambda_\Delta(n, r)$ and $h \in \mathcal{H}_\Delta(r)$. Let $\mathcal{Y}_{\lambda,m} = \sum_{\substack{\alpha \in \Lambda_\Delta(n,r'), \beta \in \Lambda_\Delta(n,r'') \\ \lambda = \alpha + \beta}} \theta_{\alpha,\alpha}^{\rho^{mr'}} \otimes \theta_{\beta,\beta}^{\rho^{mr''}}$. Clearly, for $\lambda \in \Lambda_\Delta(n, r)$ we have

$$x_\lambda \mathcal{H}_\Delta(r) = \bigoplus_{\substack{\gamma \in \Lambda_\Delta(n,r'), \delta \in \Lambda_\Delta(n,r'') \\ \lambda = \gamma + \delta}} x_\gamma \mathcal{H}_\Delta(r') \otimes x_\delta \mathcal{H}_\Delta(r'').$$

By 3.2, for $\gamma \in \Lambda_\Delta(n, r')$, $\delta \in \Lambda_\Delta(n, r'')$, $h' \in \mathcal{H}_\Delta(r')$, $h'' \in \mathcal{H}_\Delta(r'')$, we have

$$\begin{aligned} \varphi_{r',r''}(\mathcal{Y}_{\lambda,m})(x_\gamma h' \otimes x_\delta h'') &= \delta_{\lambda,\gamma+\delta} x_\gamma h' T_{\rho^{mr'}} \otimes x_\delta h'' T_{\rho^{mr''}} \\ &= \delta_{\lambda,\gamma+\delta} x_\gamma h' (X_1^{-1} \cdots X_{r'}^{-1})^m \otimes x_\delta h'' (X_1^{-1} \cdots X_{r''}^{-1})^m \\ &= \delta_{\lambda,\gamma+\delta} (x_\gamma h' \otimes x_\delta h'') (X_1^{-1} \cdots X_{r'}^{-1} X_{r'+1}^{-1} \cdots X_{r'+r''}^{-1})^m \\ &= \delta_{\lambda,\gamma+\delta} (x_\gamma h' \otimes x_\delta h'') T_{\rho^{mr}} = \theta_{\lambda,\lambda}^{\rho^{mr}}(x_\gamma h' \otimes x_\delta h''). \end{aligned}$$

It follows that $\varphi_{r',r''}(\Delta_{r',r''}(\theta_{\lambda,\lambda}^{\rho^{mr}})) = \theta_{\lambda,\lambda}^{\rho^{mr}} = \varphi_{r',r''}(\mathcal{Y}_{\lambda,m})$ and hence $\Delta_{r',r''}(\theta_{\lambda,\lambda}^{\rho^{mr}}) = \mathcal{Y}_{\lambda,m}$. \square

For $A \in \Theta_\Delta(n)$ let $\text{ro}(A) = (\sum_{j \in \mathbb{Z}} a_{i,j})_{i \in \mathbb{Z}}$ and $\text{co}(A) = (\sum_{i \in \mathbb{Z}} a_{i,j})_{j \in \mathbb{Z}}$. By [3, 7.7(2) and 7.9] we have the following result.

Lemma 3.4. *For $A \in \Theta_\Delta^+(n)$ and $\lambda \in \Lambda_\Delta(n, r)$, we have $\zeta_r(\theta_A^+)[\text{diag}(\lambda)] = \theta_{A+\text{diag}(\lambda-\text{co}(A)),r}$ if $\lambda - \text{co}(A) \in \mathbb{N}_\Delta^n$, and $\zeta_r(\theta_A^+)[\text{diag}(\lambda)] = 0$ otherwise.*

For $\lambda, \mu \in \mathbb{Z}_\Delta^n$ let $\langle \lambda, \mu \rangle = \sum_{1 \leq i \leq n} \lambda_i \mu_i - \sum_{1 \leq i \leq n} \lambda_i \mu_{i+1}$ for $\lambda, \mu \in \mathbb{Z}_\Delta^n$. We now use 3.3 and 3.4 to prove the following formula.

Corollary 3.5. *Assume $r = r' + r''$ with $r', r'' \in \mathbb{N}$. Let $A \in \Theta_\Delta^+(n)$ and $\lambda \in \Lambda_\Delta(n, r)$ with $\lambda - \text{co}(A) \in \mathbb{N}_\Delta^n$. Then we have*

$$\begin{aligned} &\Delta_{r',r''}(\theta_{A+\text{diag}(\lambda-\text{co}(A)),r}) \\ &= \sum_{\substack{B,C \in \Theta_\Delta^+(n), \mathbf{d}(A) = \mathbf{d}(B) + \mathbf{d}(C) \\ \alpha \in \Lambda_\Delta(n,r'), \beta \in \Lambda_\Delta(n,r''), \alpha + \beta = \lambda}} \mathbf{f}_{A,B,C} v^{(\mathbf{d}(B), \beta)} \theta_{B+\text{diag}(\alpha-\text{co}(B)),r'} \otimes \theta_{C+\text{diag}(\beta-\text{co}(C)),r''}, \end{aligned}$$

where $\mathbf{f}_{A,B,C}$ is as given in (3.1).

Proof. By 3.4 we have $\Delta_{r',r''}(\theta_{A+\text{diag}(\lambda-\text{co}(A)),r}) = \Delta_{r',r''}(\zeta_r(\theta_A^+)) \cdot \Delta_{r',r''}([\text{diag}(\lambda)]) = ((\zeta_{r'} \otimes \zeta_{r''}) \circ \Delta(\theta_A^+)) \cdot \Delta_{r',r''}([\text{diag}(\lambda)])$. Now the assertion follows from (3.1) and 3.3. \square

For $m \in \mathbb{Z}$ there is a map

$$(3.4) \quad \eta_m : \Theta_\Delta(n) \rightarrow \Theta_\Delta(n)$$

defined by sending $A = (a_{i,j})_{i,j \in \mathbb{Z}}$ to $(a_{i,mn+j})_{i,j \in \mathbb{Z}}$. The following lemma can be easily checked (see [4]).

Lemma 3.6. *Let $m \in \mathbb{Z}$ and $A \in \Theta_\Delta(n, r)$ with $\lambda = \text{ro}(A)$ and $\mu = \text{co}(A)$.*

- (1) *If $a_{i,j} = 0$ for $1 \leq i \leq n$ and $j \leq mn$, then $\eta_k(A) \in \Theta_\Delta^+(n)$ for $k \leq m-1$.*
- (2) *We have $\theta_{A,r} \cdot \theta_{\mu,\mu}^{\rho^{mr}} = \theta_{\eta_m(A),r} = \theta_{\lambda,\lambda}^{\rho^{mr}} \cdot \theta_{A,r}$ for $m \in \mathbb{Z}$.*

Assume $N \geq n$. There is a natural injective map

$$\sim : \Theta_\Delta(n) \longrightarrow \Theta_\Delta(N), \quad A = (a_{i,j}) \longmapsto \tilde{A} = (\tilde{a}_{i,j}),$$

where $\tilde{A} = (\tilde{a}_{i,j})$ is defined by

$$\tilde{a}_{k,l+mN} = \begin{cases} a_{k,l+mn}, & \text{if } 1 \leq k, l \leq n; \\ 0, & \text{if either } n < k \leq N \text{ or } n < l \leq N \end{cases}$$

for $m \in \mathbb{Z}$. Note that the map $\sim : \Theta_\Delta(n) \longrightarrow \Theta_\Delta(N)$ induces a map from $\Theta_\Delta^+(n)$ to $\Theta_\Delta^+(N)$. It is easy to see that there is an injective algebra homomorphism (not sending 1 to 1)

$$\iota_{n,N} : \mathcal{S}_\Delta(n, r) \longrightarrow \mathcal{S}_\Delta(N, r), \quad [A] \longmapsto [\tilde{A}] \text{ for } A \in \Theta_\Delta(n, r)$$

(see [1, §4.1]). Let $\Theta_\Delta(n, r)^{\text{ap}} = \Theta_\Delta(n)^{\text{ap}} \cap \Theta_\Delta(n, r)$. One can easily prove the following results (see [4]).

Lemma 3.7. *Assume $N > n$. Then for $A \in \Theta_\Delta(n, r)$ we have $\tilde{A} \in \Theta_\Delta(N, r)^{\text{ap}}$ and $\iota_{n,N}(\theta_{A,r}) = \theta_{\tilde{A},r}$.*

We now give a precise relation between the structure constants of the comultiplication with respect to the canonical basis $\mathbf{B}(n, r)$ of $\mathcal{S}_\Delta(n, r)$ and that with respect to the canonical basis $\mathbf{B}(N)^{\text{ap}}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$.

Theorem 3.8. *Assume $N \geq n$ and $r = r' + r''$. Let $A \in \Theta_\Delta(n, r)$, $B \in \Theta_\Delta(n, r')$, $C \in \Theta_\Delta(n, r'')$.*

- (1) *For $X \in \Theta_\Delta(N, r')$, $Y \in \Theta_\Delta(N, r'')$, we have*

$$\mathbf{g}_{\widetilde{\eta_k(A)}, X, Y}^{r', r''} = \begin{cases} \mathbf{g}_{A, L, M}^{r', r''} & \text{if } X = \widetilde{\eta_k(L)} \text{ and } Y = \widetilde{\eta_k(M)} \text{ for some } L \in \Theta_\Delta(n, r') \text{ and } M \in \Theta_\Delta(n, r''), \\ 0 & \text{otherwise} \end{cases}$$

for $k \in \mathbb{Z}$, where $\mathbf{g}_{A, L, M}^{r', r''}$ is as given in (3.3).

- (2) *If $N > n$, then there exist $k_0 \in \mathbb{Z}$ such that for $k \leq k_0$, $\widetilde{\eta_k(A)}, \widetilde{\eta_k(B)}, \widetilde{\eta_k(C)} \in \Theta_\Delta^+(N)^{\text{ap}}$ and*

$$\mathbf{g}_{A, B, C}^{r', r''} = v^{\langle \mathbf{d}(\widetilde{\eta_k(B)}), \text{co}(\widetilde{\eta_k(C)}) \rangle} \mathbf{f}_{\widetilde{\eta_k(A)}, \widetilde{\eta_k(B)}, \widetilde{\eta_k(C)}} \in \mathbb{N}[v, v^{-1}],$$

where $\widetilde{f_{\eta_k(A), \eta_k(B), \eta_k(C)}}$ is as given in (3.1).

Proof. Let $\mu = \text{co}(A)$. Then by 3.6(2) and 3.3 we have

$$\Delta_{r', r''}(\theta_{\eta_k(A), r}) = \Delta_{r', r''}(\theta_{A, r}) \Delta_{r', r''}(\theta_{\mu, \mu}^{\rho_{k_r}}) = \sum_{\substack{X \in \Theta_{\Delta}(n, r') \\ Y \in \Theta_{\Delta}(n, r'')}} g_{A, X, Y}^{r', r''} \theta_{\eta_k(X), r'} \otimes \theta_{\eta_k(Y), r''}.$$

Clearly we have $\Delta_{r', r''} \circ \iota_{n, N} = (\iota_{n, N} \otimes \iota_{n, N}) \circ \Delta_{r', r''}$. Thus by 3.7 we have

$$\Delta_{r', r''}(\widetilde{\theta_{\eta_k(A), r}}) = (\iota_{n, N} \otimes \iota_{n, N})(\Delta_{r', r''}(\theta_{\eta_k(A), r})) = \sum_{\substack{X \in \Theta_{\Delta}(n, r') \\ Y \in \Theta_{\Delta}(n, r'')}} g_{A, X, Y}^{r', r''} \widetilde{\theta_{\eta_k(X), r'}} \otimes \widetilde{\theta_{\eta_k(Y), r''}}.$$

The statement (1) follows. The statement (2) follows from (1), 3.1, 3.5 and 3.6(1). \square

Corollary 3.9. Assume $N > n$. For $A, B, C \in \Theta_{\Delta}^+(n)$ we have

$$f_{A, B, C} = v^{-\langle \mathbf{d}(B), \text{co}(C) \rangle + \langle \mathbf{d}(\tilde{B}), \text{co}(\tilde{C}) \rangle} f_{\tilde{A}, \tilde{B}, \tilde{C}} \in \mathbb{N}[v, v^{-1}],$$

where $f_{A, B, C}$ is as given in (3.1).

Proof. There exist $\mathbf{x}, \mathbf{y} \in \mathbb{N}_{\Delta}^n$ such that $\mathbf{x} + \text{co}(A) = \mathbf{y} + \text{co}(B) + \text{co}(C)$. Let $r' = \sigma(B) + \sigma(\mathbf{y})$ and $r'' = \sigma(C)$. By 3.5, 3.8(1) and 3.1 we have $f_{A, B, C} = v^{-\langle \mathbf{d}(B), \text{co}(C) \rangle} g_{A+\text{diag}(\mathbf{x}), B+\text{diag}(\mathbf{y}), C}^{r', r''} = v^{-\langle \mathbf{d}(B), \text{co}(C) \rangle} g_{\tilde{A}+\text{diag}(\mathbf{x}), \tilde{B}+\text{diag}(\mathbf{y}), \tilde{C}}^{r', r''} = v^{-\langle \mathbf{d}(B), \text{co}(C) \rangle + \langle \mathbf{d}(\tilde{B}), \text{co}(\tilde{C}) \rangle} f_{\tilde{A}, \tilde{B}, \tilde{C}} \in \mathbb{N}[v, v^{-1}].$ \square

4. THE CONNECTION BETWEEN $\dot{\mathbf{B}}(n)$ AND $\mathbf{B}(N)^{\text{ap}}$

Let X be the quotient of \mathbb{Z}_{Δ}^n by the subgroup generated by the element $\mathbf{1}$, where $\mathbf{1}_i = 1$ for all i . For $\lambda \in \mathbb{Z}_{\Delta}^n$ let $\bar{\lambda} \in X$ be the image of λ in X . Let $Y = \{\mu \in \mathbb{Z}_{\Delta}^n \mid \sum_{1 \leq i \leq n} \mu_i = 0\}$. For $\bar{\lambda} \in X$ and $\mu \in Y$ we set $\mu \cdot \bar{\lambda} = \sum_{1 \leq i \leq n} \lambda_i \mu_i$. For $\bar{\lambda}, \bar{\mu} \in X$ we set ${}_{\bar{\lambda}}\mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n)_{\bar{\mu}} = \mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n) / {}_{\bar{\lambda}}I_{\bar{\mu}}$, where ${}_{\bar{\lambda}}I_{\bar{\mu}} = \sum_{\mathbf{j} \in Y} (K^{\mathbf{j}} - v^{\mathbf{j} \cdot \bar{\lambda}}) \mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n) + \sum_{\mathbf{j} \in Y} \mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n) (K^{\mathbf{j}} - v^{\mathbf{j} \cdot \bar{\mu}})$ and $K^{\mathbf{j}} = \prod_{1 \leq i \leq n} K_i^{j_i}$. Let

$$\dot{\mathbf{U}}(\widehat{\mathbf{s}}\mathbf{l}_n) := \bigoplus_{\bar{\lambda}, \bar{\mu} \in X} {}_{\bar{\lambda}}\mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n)_{\bar{\mu}}.$$

There is a natural algebra structure on $\dot{\mathbf{U}}(\widehat{\mathbf{s}}\mathbf{l}_n)$ inherited from that of $\mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n)$ (see [12, 23.1.1]).

Let $\pi_{\bar{\lambda}, \bar{\mu}} : \mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n) \rightarrow {}_{\bar{\lambda}}\mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n)_{\bar{\mu}}$ be the canonical projection. For $\bar{\lambda} \in X$ let $1_{\bar{\lambda}} = \pi_{\bar{\lambda}, \bar{\lambda}}(1)$. The map ζ_r defined in §2 induces an algebra homomorphism

$$\dot{\zeta}_r : \dot{\mathbf{U}}(\widehat{\mathbf{s}}\mathbf{l}_n) \rightarrow \mathcal{S}_{\Delta}(n, r)$$

such that $\dot{\zeta}_r(\pi_{\bar{\lambda}, \bar{\mu}}(u)) = \dot{\zeta}_r(1_{\bar{\lambda}})\zeta_r(u)\dot{\zeta}_r(1_{\bar{\mu}})$ for $u \in \mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n)$ and $\bar{\lambda}, \bar{\mu} \in X$, and $\dot{\zeta}_r(1_{\bar{\lambda}}) = [\text{diag}(\alpha)]$ if $\bar{\lambda} = \bar{\alpha}$ for some $\alpha \in \Lambda_{\Delta}(n, r)$, $\dot{\zeta}_r(1_{\bar{\lambda}}) = 0$ otherwise.

For $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta} \in X$, there is a well-defined linear map

$$\Delta_{\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}} : \bar{\alpha} + \bar{\gamma} \mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n)_{\bar{\beta} + \bar{\delta}} \rightarrow \bar{\alpha} \mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n)_{\bar{\beta}} \otimes \bar{\gamma} \mathbf{U}(\widehat{\mathbf{s}}\mathbf{l}_n)_{\bar{\delta}}$$

such that $\Delta_{\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}}(\pi_{\bar{\alpha}+\bar{\gamma}, \bar{\beta}+\bar{\delta}}(x)) = (\pi_{\bar{\alpha}, \bar{\beta}} \otimes \pi_{\bar{\gamma}, \bar{\delta}})(\Delta(x))$ for $x \in \mathbf{U}(\widehat{\mathfrak{sl}}_n)$. This collection of maps is called the comultiplication of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ (see [12]).

Let $\dot{\mathbf{B}}(n)$ be the canonical basis of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ defined by Lusztig [12]. Let $\mathcal{Y}(n) = \{A \in \Theta_\Delta(n)^{\text{ap}}, A - E \notin \Theta_\Delta(n)\}$. By the proof of [14, 4.3], we see that for $A \in \mathcal{Y}(n)$, there exists a unique $\mathbf{b}_A \in \dot{\mathbf{B}}(n)$ such that $\dot{\zeta}_r(\mathbf{b}_A) = \theta_{A+mE, r}$ if $r = \sigma(A) + mn$ for some $m \geq 0$, and $\dot{\zeta}_r(\mathbf{b}_A) = 0$ otherwise. Furthermore we have $\dot{\mathbf{B}}(n) = \{\mathbf{b}_A \mid A \in \mathcal{Y}(n)\}$ (cf. [17, 15]).

For $\lambda, \mu \in \Lambda_\Delta(n, r)$ let ${}_\lambda \Theta_\Delta(n, r)_\mu = \{A \in \Theta_\Delta(n, r) \mid \text{ro}(A) = \lambda, \text{co}(A) = \mu\}$. For $\bar{\lambda}, \bar{\mu} \in X$ let ${}_{\bar{\lambda}} \mathcal{Y}(n)_{\bar{\mu}} = \{A \in \mathcal{Y}(n) \mid \overline{\text{ro}(A)} = \bar{\lambda}, \overline{\text{co}(A)} = \bar{\mu}\}$. Then for $A \in {}_{\bar{\lambda}} \mathcal{Y}(n)_{\bar{\mu}}$ we have $\mathbf{b}_A \in {}_{\bar{\lambda}} \mathbf{U}(\widehat{\mathfrak{sl}}_n)_{\bar{\mu}}$. For $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta} \in X$ and $A \in {}_{\bar{\alpha}+\bar{\gamma}} \mathcal{Y}(n)_{\bar{\beta}+\bar{\delta}}$, we write

$$(4.1) \quad \Delta_{\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}}(\mathbf{b}_A) = \sum_{\substack{B \in {}_{\bar{\alpha}} \mathcal{Y}(n)_{\bar{\beta}} \\ C \in {}_{\bar{\gamma}} \mathcal{Y}(n)_{\bar{\delta}}}} \mathbf{h}_{A, B, C} \mathbf{b}_B \otimes \mathbf{b}_C$$

where $\mathbf{h}_{A, B, C} \in \mathcal{Z}$.

Finally, we give a precise relation between the structure constants of the comultiplication with respect to the canonical basis $\dot{\mathbf{B}}(n)$ of $\dot{\mathbf{U}}(\widehat{\mathfrak{sl}}_n)$ and that with respect to the canonical basis $\mathbf{B}(N)^{\text{ap}}$ of $\mathbf{U}(\widehat{\mathfrak{sl}}_N)^+$.

Theorem 4.1. *Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta} \in X$, $A \in {}_{\bar{\alpha}+\bar{\gamma}} \mathcal{Y}(n)_{\bar{\beta}+\bar{\delta}}$, $B \in {}_{\bar{\alpha}} \mathcal{Y}(n)_{\bar{\beta}}$ and $C \in {}_{\bar{\gamma}} \mathcal{Y}(n)_{\bar{\delta}}$. Let $r' = \sigma(B)$, $r'' = \sigma(C)$. Assume $N > n$ and $\mathbf{h}_{A, B, C} \neq 0$. Then there exist $m \in \mathbb{N}$ and $k_0 \in \mathbb{Z}$ such that $\sigma(B) + \sigma(C) = \sigma(A) + mn$, $\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k \in \Theta_\Delta^+(N)^{\text{ap}}$ and*

$$\mathbf{h}_{A, B, C} = \mathbf{g}_{A+mE, B, C}^{r', r''} = v^{\langle \mathbf{d}(\widetilde{B}_k), \text{co}(\widetilde{C}_k) \rangle} \mathbf{f}_{\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k} \in \mathbb{N}[v, v^{-1}],$$

for $k \leq k_0$, where $A_k = \eta_k(A + mE)$, $B_k = \eta_k(B)$, $C_k = \eta_k(C)$, $\mathbf{f}_{\widetilde{A}_k, \widetilde{B}_k, \widetilde{C}_k}$ is as given in (3.1) and $\mathbf{g}_{A+mE, B, C}^{r', r''}$ is as given in (3.3).

Proof. Note that we have $\dot{\zeta}_{r'}(\mathbf{b}_B) = \theta_{B, r'}$ and $\dot{\zeta}_{r''}(\mathbf{b}_C) = \theta_{C, r''}$. Let $r = r' + r''$. Since $\dot{\zeta}_r(\mathbf{b}_A) \neq 0$, we conclude that $r = \sigma(A) + mn$ for some $m \in \mathbb{N}$ and $\dot{\zeta}_r(\mathbf{b}_A) = \theta_{A+mE, r}$. Clearly we have $(\dot{\zeta}_{r'} \otimes \dot{\zeta}_{r''}) \circ \Delta_{\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}}(\mathbf{b}_A) = ([\text{diag}(\text{ro}(B))] \otimes [\text{diag}(\text{ro}(C))]) \cdot \Delta_{r', r''}(\dot{\zeta}_r(\mathbf{b}_A)) \cdot ([\text{diag}(\text{co}(B))] \otimes [\text{diag}(\text{co}(C))])$. This together with 3.3 and 4.1 implies that

$$\sum_{\substack{A' \in {}_{\bar{\alpha}} \mathcal{Y}(n)_{\bar{\beta}} \\ A'' \in {}_{\bar{\gamma}} \mathcal{Y}(n)_{\bar{\delta}}}} \mathbf{h}_{A, A', A''} \dot{\zeta}_{r'}(\mathbf{b}_{A'}) \otimes \dot{\zeta}_{r''}(\mathbf{b}_{A''}) = \sum_{\substack{A' \in {}_{\text{ro}(B)} \Theta_\Delta(n, r')_{\text{co}(B)} \\ A'' \in {}_{\text{ro}(C)} \Theta_\Delta(n, r'')_{\text{co}(C)}}} \mathbf{g}_{A+mE, A', A''}^{r', r''} \theta_{A', r'} \otimes \theta_{A'', r''}.$$

It follows that $\mathbf{h}_{A, B, C} = \mathbf{g}_{A+mE, B, C}^{r', r''}$. Now the result follows from 3.8(2). \square

REFERENCES

- [1] B. Deng, J. Du and Q. Fu, *A double Hall algebra approach to affine quantum Schur–Weyl theory*, London Mathematical Society Lecture Note Series, **401**, Cambridge University Press, 2012.
- [2] J. Du and Q. Fu, *A modified BLM approach to quantum affine \mathfrak{gl}_n* , Math. Z. **266** (2010), 747–781.

- [3] J. Du and Q. Fu, *The Integral quantum loop algebra of \mathfrak{gl}_n* , preprint, arXiv:1404.5679.
- [4] Q. Fu and T. Shoji, *Positivity properties for canonical bases of modified quantum affine \mathfrak{sl}_n* , preprint, arXiv:1407.4228.
- [5] V. Ginzburg and E. Vasserot, *Langlands reciprocity for affine quantum groups of type A_n* , Internat. Math. Res. Notices 1993, 67–85.
- [6] R. M. Green, *The affine q -Schur algebra*, J. Algebra **215** (1999), 379–411.
- [7] I. Grojnowski, *The coproduct for quantum GL_n* , preprint.
- [8] H. Jakobsen and H. Zhang, *The Exponential Nature and Positivity*, Algebr. Represent. Theory **9** (2006), 267–284.
- [9] M. Kashiwara, *On crystal bases of the q -analogue of universal enveloping algebras*, Duke Math. J. **63** (1991), 465–516.
- [10] G. Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. **4** (1991), 365–421.
- [11] G. Lusztig, *Affine quivers and canonical bases*, Inst. Hautes Études Sci. Publ. Math. **76** (1992), 111–163.
- [12] G. Lusztig, *Introduction to quantum groups*, Progress in Math. **110**, Birkhäuser, 1993.
- [13] G. Lusztig, *Aperiodicity in quantum affine \mathfrak{gl}_n* , Asian J. Math. **3** (1999), 147–177.
- [14] G. Lusztig, *Transfer maps for quantum affine \mathfrak{sl}_n* , in: Representations and quantizations (Shanghai, 1998), China High. Educ. Press, Beijing, 2000, 341–356.
- [15] K. McGerty, *On the geometric realization of the inner product and canonical basis for quantum affine \mathfrak{sl}_n* , Algebra Number Theory **6** (2012), 1097–1131.
- [16] C. M. Ringel, *The composition algebra of a cyclic quiver*, Proc. London Math. Soc. **66** (1993), 507–537.
- [17] O. Schiffmann and E. Vasserot, *Geometric construction of the global base of the quantum modified algebra of $\widehat{\mathfrak{gl}}_n$* , Transform. Groups **5** (2000), 351–360.
- [18] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. **100** (1999), 267–297.

DEPARTMENT OF MATHEMATICS, TONGJI UNIVERSITY, SHANGHAI, 200092, CHINA.

E-mail address: q.fu@hotmail.com, q.fu@tongji.edu.cn